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Section 5.2: Series

Def: An (infinite) series is an expression of the form

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots$$

Note: There are two sequences to consider given a series $\sum_{n=0}^{\infty} a_n$.

(a) The sequence of terms / summands, a_n

(b) The sequence of partial sums, $S_k = \sum_{n=0}^{k} a_n$

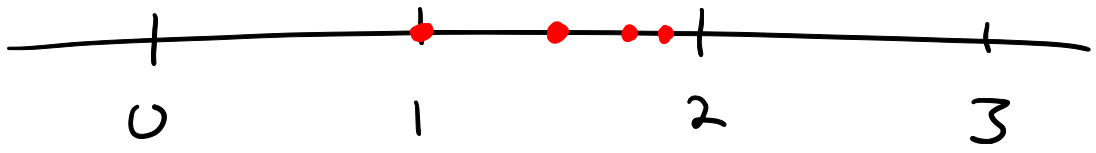
$$S_0 = a_0$$
$$S_1 = a_0 + a_1$$
$$S_2 = a_0 + a_1 + a_2$$
$$S_3 = a_0 + a_1 + a_2 + a_3$$
$$\vdots$$
$$S_k = a_0 + a_1 + \cdots + a_k$$

Def: If the sequence of partial sums converges to a real number S, then the infinite series converges. In this case, S is called the sum of the series, and we write

$$\sum_{n=0}^{\infty} a_n = S$$

If the sequence of partial sums diverges, we say that the series diverges.

Ex 1 Draw a number line from 0 to 3 below.



(a) On the number line, place a point at 1. Then $1+\frac{1}{2}$. Then $1+\frac{1}{2}+\frac{1}{4}$. Then $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$.

(b) If you were to continue the process indefinitely, to what number would the points that you're drawing keep getting closer? Why?

Closer to 2 (not reaching)

→ b/c halving distance to 2 every time

(c) Write $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^k}$ in sigma notation.

$$\sum_{n=0}^{k} \frac{1}{2^n}$$

(d) What is $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$?

↳ seq. of partial sums: $S_0 = 1$, $S_1 = 1+\frac{1}{2}$, $S_2 = 1+\frac{1}{2}+\frac{1}{4}$, $\vdots$, $S_k = (c)$

By (b), $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \lim_{k\to\infty} S_k = 2$

Ex 2 Suppose r is a constant and consider the series $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$. For which values of r does this series converge?

By HW 1.11: If $\sum_{n=0}^{\infty} r^n$ converges $\longrightarrow \lim_{n\to\infty} r^n = 0$

$\longrightarrow |r| < 1$

$S_0 = 1$

$S_1 = 1 + r$

$S_2 = 1 + r + r^2$

$\vdots$

$S_k = 1 + r + \cdots + r^k \longrightarrow (1-r)S_k = (1-r)(1 + r + \cdots + r^k)$

$= 1 + r + r^2 + \cdots + r^k$

$- r - r^2 - \cdots - r^k - r^{k+1}$

$= 1 - r^{k+1}$

$\longrightarrow S_k = \dfrac{1 - r^{k+1}}{1-r}$ ← b/c $r \neq 1$

$$\sum_{n=0}^{\infty} r^n = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1 - r^{k+1}}{1-r} = \frac{1}{1-r}$$

($r^{k+1} \to 0$)

$r = \frac{1}{2}$: $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - \frac{1}{2}} = 2$

For $r \geq 1$: $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots = \infty$

For $r \leq -1$: $\sum_{n=0}^{\infty} r^n$ DNE

Def: A geometric series is a series whose terms form a geometric sequence: $\sum_{n=n_0}^{\infty} ar^n$

Note: $\sum_{n=n_0}^{\infty} ar^n = ar^{n_0} + ar^{n_0+1} + \cdots = a\left(r^{n_0} + r^{n_0+1} + \cdots\right)$

$= ar^{n_0}\left(1 + r + r^2 + \cdots\right)$

$= \dfrac{ar^{n_0}}{1-r}$

Ex 3 Find the exact value of the sum

$$10 - \frac{10}{9} + \frac{10}{81} - \frac{10}{729} + \frac{10}{6561} - \cdots$$

$10\left(1 - \frac{1}{9} + \frac{1}{81} - \frac{1}{729} + \cdots\right)$

$10\left(\left(\frac{-1}{9}\right)^0 + \left(-\frac{1}{9}\right)^1 + \left(\frac{-1}{9}\right)^2 + \left(-\frac{1}{9}\right)^3 + \cdots\right)$

$10\left(\frac{1}{1-\left(\frac{-1}{9}\right)}\right) = 10\left(\frac{9}{10}\right) = 9$

Ex 4 Write the number $5.\overline{279} = 5.279279279\ldots$ as a ratio of two integers.

$$5 + \frac{279}{1000} + \frac{279}{1000000} + \frac{279}{1000000000} + \cdots$$

$$5 + \frac{279}{1000} + \frac{279}{1000^2} + \frac{279}{(1000)^3} + \cdots$$

$$5 + \frac{279}{1000}\left(1 + \frac{1}{1000} + \frac{1}{1000^2} + \cdots\right)$$

$$5 + \frac{279}{1000}\left(\frac{1}{1 - \frac{1}{1000}}\right)$$

$$5 + \frac{279}{999}$$

$$\frac{5274}{999}$$

Ex 5 The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the *harmonic series*. Does it converge or diverge?

$\frac{1}{n}$ — not geometric seq. so $\sum_{n=1}^{\infty} \frac{1}{n}$ not geo. series

$S_1 = 1$ $> \frac{1}{2}$

$S_2 = 1 + \frac{1}{2}$

$S_3 = 1 + \frac{1}{2} + \frac{1}{3}$ $> \frac{1}{2} + \frac{1}{2}$
$> \frac{1}{2} \quad > \frac{1}{4} \quad > \frac{1}{4}$

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

$S_7 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$ $> \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$
$> \frac{1}{2} \quad > \frac{1}{4} \quad > \frac{1}{4} \quad > \frac{1}{8} \quad > \frac{1}{8} \quad > \frac{1}{8} \quad > \frac{1}{8}$

$S_{2^k - 1} > \frac{k}{2}$ ⟶ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Thm: (Algebraic Properties of Convergent Series) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be Convergent series. Then the following algebraic properties hold:

(a) The series $\sum_{n=1}^{\infty}(a_n + b_n)$ converges and $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

(b) The series $\sum_{n=1}^{\infty}(a_n - b_n)$ converges and $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

(c) For any real number c, the series $\sum_{n=1}^{\infty} ca_n$ converges and $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$.

Ex 6 Does $\sum_{n=1}^{\infty}\left(2^{-\frac{1}{n}}-2^{-\frac{1}{n+1}}\right)$ converge or diverge? If it converges, to what does it converge?

$$S_1 = 2^{-1} - 2^{-1/2}$$

$$S_2 = \left(2^{-1} - 2^{-1/2}\right) + \left(2^{-1/2} - 2^{-1/3}\right)$$

$$S_3 = \left(2^{-1} - 2^{-1/2}\right) + \left(2^{-1/2} - 2^{-1/3}\right) + \left(2^{-1/3} - 2^{-1/4}\right)$$

$$\vdots$$

$$S_k = \left(2^{-1} - 2^{-1/2}\right) + \left(2^{-1/2} - 2^{-1/3}\right) + \cdots + \left(2^{-1/k} - 2^{-1/(k+1)}\right)$$

$$= 2^{-1} - 2^{-1/(k+1)}$$

$$\sum_{n=1}^{\infty} 2^{-1/n} - 2^{-1/(n+1)} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} 2^{-1} - 2^{-1/(k+1)} = 2^{-1} - 1 = -\frac{1}{2}$$

Q. What are the partial sums of the series $\sum_{n=0}^{\infty} t_n - t_{n+1}$?

$$S_0 = t_0 - t_1$$

$$S_1 = (t_0 - t_1) + (t_1 - t_2) = t_0 - t_2$$

$$S_2 = (t_0 - t_1) + (t_1 - t_2) + (t_2 - t_3) = t_0 - t_3$$

$$\vdots$$

$$S_k = t_0 - t_{k+1}$$

Def: Suppose that $(t_n)_{n=0}^{\infty}$ is a convergent sequence with $\lim_{n\to\infty} t_n = L$. Then the series

$$\sum_{n=n_0}^{\infty} t_n - t_{n+1}$$

is called a telescoping series. It converges to $t_{n_0} - L$.

Ex 7 Compute $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

PFD: $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$= \frac{1}{1} - \frac{1}{2} + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

$$= 1 - \lim_{k \to \infty} \frac{1}{k}$$

$$= 1$$

Q. How common are geometric and telescoping series? Will we be able to compute the values of very many sums?